Optimal Monetary Policy under Incomplete Markets and Aggregate Uncertainty *†

Oleksiy Kryvtsov
Bank of Canada

Malik Shukayev
Bank of Canada

Alexander Ueberfeldt
Bank of Canada

April 2008

Abstract

This paper examines the role of monetary policy in an environment with aggregate risk and incomplete markets. In a two-period overlapping-generations model with aggregate uncertainty and nominal bonds, optimal monetary policy attains the ex-ante Pareto optimal allocation. This policy aims to stabilize the savings rate in the economy via the effect of expected inflation on real returns of nominal bonds. In the first-best monetary equilibrium with uncertainty, expected inflation is procylical and, on average, higher than without uncertainty. Optimal monetary policy is close to inflation targeting if the persistence of income is high, and close to price-level targeting if the persistence of income is low. The results extend to more general environments with multiple assets and infinitely-lived agents.

JEL classification: E5

Keywords: Optimal monetary policy; Inflation targeting; Price-level targeting

*Contact address: Malik Shukayev, 234 Wellington St. 5-West, Ottawa, ON K1A 0G9, Canada. Phone 613-782-8164. Fax: 613-782-7163. E-mail: mshukayev@bankofcanada.ca

†We thank Robert Amano, Steve Ambler, Pierre Duguay, Mario Crucini, John Murray, Peter Ireland, Larry Jones, Sharon Kozicki, seminar participants at the Bank of Canada, Higher School of Economics (Moscow), National Bank of Kazakhstan, Queen's University, and Université de Montréal for their comments and suggestions.

1. Introduction

What is the role of monetary policy in an environment with aggregate risk and incomplete asset markets? We study a standard two-period overlapping-generations model (OLG) with aggregate-endowment uncertainty and find that monetary policy can achieve the ex-ante Pareto optimal allocation. The optimal monetary policy that implements the first-best allocation stabilizes savings rates by affecting the expected real return on nominal bonds. It is characterized by expected inflation that on average is higher than without uncertainty, a positive correlation between expected inflation and income, an inverse relationship between the volatility of expected inflation and income persistence, and a nonstationary price level.

The intuition for the properties of optimal monetary policy is based on the link between individual savings decisions and the optimal allocation of consumption across generations. Thus, to understand the role of these properties of the optimal policy, it helps to first restate the main findings of the literature on individual savings behavior under uncertainty.[1] When faced with uninsurable endowment risk and a constant rate of return on savings, risk averse individuals smooth their consumption by varying their savings rates with income. When current income is higher than expected future income, individuals save a larger fraction of income to move part of the current "windfall" to the future. When current income is lower than expected future income individuals save less, taking advantage of the anticipated increase in future income. Furthermore, for precautionary reasons, on average, they save more than optimal without uncertainty, or under complete asset markets. The responsiveness of the savings rate to income fluctuations depends on the persistence of income disturbances. When income fluctuations are long-lived, incentives to vary savings are weak, whereas when income movements are transitory, incentives to vary savings are strong.

In our model, the young individuals face uncertainty regarding the rate of return on their savings in nominal assets. The return depends on the realization of the endowment of the young next period, and on monetary policy. The individual savings behavior of the young maps directly into the allocation of goods between the young and the old because it determines the price of nominal assets sold by the old to the young. With nominal bonds, monetary policy affects savings via its effect on expected inflation. Without higher average inflation, risk averse individuals tend to save more than optimal for precautionary reasons. Higher average inflation serves as an optimal tax on savings, which discourages oversaving. Turning to the second property of optimal policy, the first-best consumption allocation calls

[1] The papers on this topic are too numerous to be mentioned and fall into "income fluctuations problem" and "permanent income hypothesis" research agendas.

for a constant savings rate. Positive correlation between expected inflation and income implies that when income is high (low) the expected tax on savings is higher (lower) too. This discourages individuals from varying savings rates to smooth consumption over time and thereby stabilizes the savings rate. The third property is a consequence of the fact that with higher income persistence individuals have less expected variation in income, and therefore less incentive to vary their savings rates. As a result, optimal expected inflation has to vary less to discourage consumption smoothing. In the limit, when the income process is random walk, optimal expected inflation is constant. This implies the inverse relationship between volatility of expected inflation and persistence of income. Finally, a nonstationary price level is a result of optimal monetary policy being completely forward looking and optimal expected inflation being independent of the current or past price levels. It is important to stress that all of these four properties are optimal responses to aggregate uncertainty. Without uncertainty, the optimal expected inflation is always zero, and the optimal price level is constant.

We apply our analysis of the *unconstrained* optimal monetary policy to policies that are *constrained* by publicly observable variables, or "targets." Such constrained policies are often preferred to the unconstrained ones due to the uncertainty about economic mechanisms in the real economy, or uncertainty associated with data revisions. Another potential advantage of using constrained policies is alleviation of the "inflation bias" that stems from the time-inconsistency problem faced by the monetary authority.[2] We will focus on two particular targeting policies: inflation targeting (IT) and price-level targeting (PT). Starting in the 1990s, several central banks announced inflation targeting as their monetary policy framework.[3] More recently, price-level targeting has been proposed as an alternative to inflation targeting.[4] The debate focuses on whether a policy that stabilizes the price-level (possibly around a deterministic trend) is preferable to policies that stabilize the inflation rate. Under inflation targeting, the monetary authority announces a desired level of annual inflation and conducts monetary policy in a way consistent with that objective.[5] Due to shocks and an imperfect degree of control over prices, the actual inflation rate may deviate from the desired

[2]See Walsh (1998) and Woodford (2003) for reviews of constrained monetary policy problems.

[3]Ball and Sheridan (2003) provide a list of central banks that adopted inflation targeting, as well as timing details and performance evaluations for this policy change.

[4]There has been a series of policy oriented papers that discuss price-level targeting as an alternative to inflation targeting. To name a few, Carlstrom and Fuerst (2002), Dittmar, Gavin and Kydland (1999), Dittmar and Gavin (2005), Vestin (2000), Gavin and Stockman (1991), Duguay (1994), Ortega and Rebei (2006), Svensson (1999), Smets (2003), Yetman (2005).

[5]As the Bank of Canada states: "monetary policy needs to aim at the 2 per cent target midpoint over the six to eight quarters that are required for monetary policy to have most of its effect. By consistently aiming at 2 per cent for the 12-month rate of inflation, monetary policy can enhance the predictability of average inflation over longer time horizons."

level. For this reason, policies aiming at stabilizing the *inflation level* do not necessarily imply a stationary *price level*. Specifically, to stabilize the level of inflation, the monetary authority does not have to react to past deviations from a predetermined price level. The price level may drift arbitrarily far away from any predetermined path, as the deviations accumulate over time. Conversely, to keep the price level close to the predetermined trend, the central bank must reverse deviations from the desired inflation level. Hence, under inflation targeting, transient deviations of the price level are not offset by the monetary authority and so the price level is nonstationary. It is (trend-)stationary under price-level targeting, when the price level, as opposed to the inflation level, is controlled.

We analyze the welfare implications of IT and PT in our framework and find that the welfare rankings of inflation targeting and price level targeting depend crucially on the persistence of income. If income is close to a random walk, inflation targeting is close to the optimal monetary policy, as IT implies stable expected inflation. Conversely, if income fluctuations are transitory, price-level targeting is close to the optimal monetary policy since PT implies a positive correlation between expected inflation and income. Specifically, when income is high, a low realized price level calls for higher expected inflation in order to return the price level back to the trend.

We summarize that the optimal monetary policy combines features inherent to inflation targeting (nonstationary price level) and price-level targeting (reversion of price-level deviations). The balance of these features in the optimal monetary policy depends on the persistence of aggregate income.

These results can be generalized to more complicated settings. In a natural extension of the benchmark model, we add a productive real asset, land, to the economy. Land is combined with the labor endowment of young individuals to produce consumption goods. The young are required to buy land for money before they can produce. They borrow money from the old individuals via noncontingent nominal bonds. These nominal bonds dominate money in terms of the rate of return and can be interpreted as mortgage contracts. In this richer model money is held not because of its store-of-value properties, but because it facilitates land purchases. As a result, monetary policy affects allocations primarily via its effect on the bond prices. Despite this richer structure, the same qualitative results are obtained for optimal and simple targeting regimes of monetary policy. Thus, our results are robust to the introduction of extra assets.

We also demonstrate that our results do not depend on the assumption of a two period life span or an OLG structure. To do this we consider a simple infinite-horizon environment, in which there is aggregate income risk, ex-post income heterogeneity across agents, and market incompleteness due to uncontingent nominal asset being the only means of savings. We show that the monetary equilibrium in the infinite-horizon model is equivalent to that in the OLG economy. A necessary assumption for this result is ex-post income heterogeneity across agents and time. It creates demand for trade that facilitates risk sharing. We conclude from our two extensions that the main properties of the optimal monetary policy under aggregate uncertainty and incomplete markets are robust and extend beyond the limits of the OLG structure.[6]

The paper contributes to several areas of research in monetary economics. To our knowledge, this is the first paper to analyze monetary policy in a stochastic OLG environment. Perhaps surprisingly, previous research on monetary policy in OLG models focused exclusively on deterministic models. Suboptimality of positive inflation was one of the main findings of that literature.[7] In a recent paper, Akyol (2004) also finds positive optimal inflation in an environment with infinitely lived agents, who are subject to uninsurable idiosyncratic endowment risk and borrowing constraints. With no aggregate uncertainty, the price level in Akyol's model increases over time in a deterministic fashion. In our model, we provide a full characterization of optimal monetary policy under aggregate uncertainty. Finally, our paper contributes to the debate on the merits of price level stabilization. It shows that it is optimal to only partially offset transient shocks to the price level. Hence, the optimal policy combines some features of both inflation targeting (implying nonstationary price level) and price-level targeting (reverting price-level deviations from the trend).

The paper proceeds as follows. Section 2 introduces and analyzes the model with fiat money as the only asset. In Section 3, the model is extended to include productive land as well as an additional interest bearing asset. Section 4 analyzes an infinite-horizon model and shows result-equivalence with the OLG model of Section 2. Section 5 contains concluding remarks. Proofs and derivations are collected in the appendices.

[6]Fiscal policy is also capable of implementing redistributions akin to monetary transfers in our model. The design of fiscal policy in economies with valued money has to take into account distributional consequences of monetary policy emphasized in this paper.

[7]See, for example, Wallace (1992) or Champ and Freeman (2001).

2. An OLG Model With Fiat Money

In this section, we focus on a two-period overlapping-generations endowment economy in which fiat money is the only asset. The young individuals in this economy use money to save for the time when they are old. Monetary policy affects real returns on savings via its effect on expected inflation. Given asset market incompleteness, monetary policy has the potential to improve the average welfare in the economy.[8]

2.1 The Environment

There is a unit measure of identical individuals born in every period. Each generation lives for two periods. The young person born in period t is endowed with w_t units of a perishable consumption good in period t and zero units in period $t+1$. The endowment w_t is random and represents the only source of uncertainty in the model. The log of the endowment follows a first-order autoregressive process:

$$\ln w_t = \rho \ln w_{t-1} + \varepsilon_t \ ,$$

where ε_t are independent and identically distributed (i.i.d.) draws from a zero-mean normal distribution.

The single asset in the economy is fiat money supplied by the government. In period 1 there is an initial old generation that has no endowment and holds M_1 units of the money stock.[9]

The timing of events is as follows. At the beginning of period t the old generation holds M_{t-1} units of fiat money which they acquired in the previous period. Before the current endowment w_t is realized, the government prints (or destroys) money in the amount of $M_t - M_{t-1}$, and distributes it evenly among the old individuals via lump-sum transfer (or tax if negative) $T_t = M_t - M_{t-1}$. The assumption that monetary transfers occur before the realization of the current endowment, reflects the limited ability of the government policy to react to current shocks in the economy, and implies an incomplete degree of control over the price level. After the realization of the current endowment, w_t, the young agents consume

[8] Markets are incomplete for two reasons. First, the overlapping-generations structure implies that newborn individuals cannot insure against the endowment risk. Second, young individuals, who save in the form of a noncontingent asset, cannot fully insure against rate-of-return risk.

[9] In this model M can be thought of as the sum of two equivalent assets: fiat money and discount bonds. The fact that nominal interest rate on bonds is fixed at zero does not affect our results. We relax this assumption in the model of section 3, where the nominal interest rate on bonds is endogeneous.

c_t^y units of their endowment. The remaining goods, $(w_t - c_t^y)$, are exchanged for M_t^d units of money at the price P_t. Thus, a young person born in period t, solves the following problem:

$$\max u(c_t^y) + \beta E_t u(c_{t+1}^o) \tag{1}$$

subject to

$$\begin{aligned} P_t c_t^y + M_t^d &\leq P_t w_t \ , & (2) \\ P_{t+1} c_{t+1}^o &\leq M_t^d + T_{t+1} \ , & (3) \end{aligned}$$

where c_{t+1}^o is the person's consumption when old, T_t is the monetary transfer from the government in period t, and β is the discount factor. The operator E_t denotes the expected value conditional on the history of endowment realizations through the end of period t. Throughout the paper we use the following functional form for the period utility function: $u(c) = c^{1-\gamma}/(1-\gamma)$; $\gamma > 0$.

2.2 Monetary Equilibrium

Let μ_t denote the growth of money supply in the economy in period t, $\mu_t = \frac{M_t}{M_{t-1}}$, where μ_1 is fixed at 1 without a loss of generality. Monetary policy is defined as an infinite sequence of money growth rates, $\{\mu_t\}_{t=1}^{\infty}$.

Definition 1 *Given a monetary policy $\{\mu_t\}_{t=1}^{\infty}$, a monetary equilibrium for this economy is a set of prices $\{P_t\}_{t=1}^{\infty}$ and allocations $\{c_t^y, c_t^o, M_t^d\}_{t=1}^{\infty}$, such that for all $t = 1, 2, 3, ...$* [10]

- *allocations c_t^y, c_{t+1}^o and M_t^d solve the generation t's problem (1)-(3), and*
- *the goods and money market clear:*

$$\begin{aligned} c_t^y + c_t^o &= w_t \ , \\ M_t^d &= M_t \ . \end{aligned}$$

In the next two subsections we first characterizes the optimal allocation and the optimal monetary policy that implements it as a monetary equilibrium, and second, compare

[10] All of the variables in the definition are random variables conditional on histories of endowment realizations. We suppress the "state" notation for simplicity.

social welfare in equilibria with price-level and inflation targeting to that under the optimal monetary policy.

2.3 Optimal Monetary Policy

To find the optimal monetary policy, we start by defining the social welfare function and solving the social planner's problem for the optimal allocation. We then ask whether this allocation can be implemented as a monetary equilibrium.

2.3.1 The Social Planner's Problem

The social planner is assumed to treat all generations equally. Let the average (ex-post) utility over T periods be:

$$\begin{aligned} V_T &= \frac{1}{T}\left[\beta u(c_1^o) + \left[\sum_{t=1}^{T-1}\left[u(c_t^y) + \beta u(c_{t+1}^o)\right]\right] + u(c_T^y)\right] \\ &= \frac{1}{T}\sum_{t=1}^{T}\left[u(c_t^y) + \beta u(c_t^o)\right] \ . \end{aligned} \tag{4}$$

We define the social welfare function as

$$\lim_{T\to\infty} \inf E\left[V_T\right] \ . \tag{5}$$

This welfare criterion treats all generations equally by attaching the same welfare weight to the expected utility of every generation.

The social planner maximizes (5) subject to the resource constraint for all periods:

$$c_t^y + c_t^o \leq w_t, \text{ for } t = 1, 2, \ldots \ .$$

We show in Appendix A that the solution to this problem is the sequence of consumptions $\{c_t^y, c_t^o\}_{t=1}^T$ such that in each period the marginal utilities of consumption of the young and of the old are equal. This link between individual savings decisions and the optimal allocation of consumption across generations is the determinant of the properties of optimal monetary policy in our model.

$$\begin{aligned} u'\left(c_t^y\right) &= \beta u'(c_t^o) \ , \\ c_t^y + c_t^o &= w_t \ . \end{aligned}$$

For the case of a constant relative risk aversion (CRRA) period utility function, $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, the first-best allocation is:

$$c_t^y = \frac{1}{1+\beta^{\frac{1}{\gamma}}} w_t \ , \tag{6}$$

$$c_t^o = \frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}} w_t \ , \tag{7}$$

for all $t = 1, 2, \ldots$. Note that the first-best allocation calls for a constant savings rate of the young equal to $\frac{w_t - c_t^y}{w_t} = \frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}}$.

There are two reasons for using the undiscounted welfare function (4) rather than the discounted one,

$$V_T = u(c_1^o) + \sum_{t=1}^{T} \beta^{t-1} \left[u(c_t^y) + \beta u(c_{t+1}^o) \right] . \tag{8}$$

The discounted social welfare function (8) implies that the optimal consumption is equally divided between the young and the old: $c_t^y = c_t^o = \frac{1}{2} w_t$. This pattern of lifetime consumption does not maximize ex-ante utility of young individuals in an OLG economy, because they discount old-age consumption and would prefer higher expected consumption in the young age. On the contrary, the consumption allocation (6) and (7) implied by the undiscounted welfare function maximizes the unconditional expected utility of any given generation (except the initial old). Hence, it is the unique ex-ante optimal allocation.

Secondly, without endowment uncertainty, this allocation is implementable as a market equilibrium with a constant money stock. This implies that the amount of redistribution of endowment required to achieve the optimal allocation is minimal. In contrast, under the discounted social welfare function redistribution is large, even in the absence of uncertainty.

2.3.2 *Implementing the Optimal Allocation as a Monetary Equilibrium*

Suppose the first-best allocation can be implemented in a monetary equilibrium. Any monetary equilibrium must satisfy the following two necessary and sufficient (due to concavity of the problem) first-order conditions:

$$u'(w_t - \frac{M_t^d}{P_t}) = \beta E_t \left[u'(\frac{M_t^d + T_{t+1}}{P_{t+1}}) \frac{P_t}{P_{t+1}} \right], \tag{9}$$

$$\frac{P_t}{P_{t+1}} = \frac{M_t}{M_{t+1}} \frac{w_{t+1} - c_{t+1}^y}{w_t - c_t^y}. \tag{10}$$

Equation (9) is a standard intertemporal Euler condition. Equation (10) is derived from the budget constraint of the young (2) held with equality, and from the money market clearing condition $M_t^d = M_t$.

These conditions, with the first-best allocation (6) and (7), imply the expressions for money growth

$$\frac{M_{t+1}}{M_t} = E_t \left[\left(\frac{w_{t+1}}{w_t} \right)^{1-\gamma} \right], \tag{11}$$

and for prices

$$\frac{P_{t+1}}{P_t} = \frac{M_{t+1}}{M_t} \frac{w_t}{w_{t+1}} = E_t \left[\left(\frac{w_{t+1}}{w_t} \right)^{1-\gamma} \right] \frac{w_t}{w_{t+1}} . \tag{12}$$

Given the assumption of log normality of the endowment process, equations (11) and (12) imply

$$m_{t+1} - m_t = \frac{(1-\gamma)^2 \sigma^2}{2} + (1-\rho)(\gamma - 1)\omega_t, \tag{13}$$

$$p_{t+1} - p_t = \frac{(1-\gamma)^2 \sigma^2}{2} + \gamma(1-\rho)\omega_t - \varepsilon_{t+1} , \tag{14}$$

where $m_t \equiv \ln M_t$, $p_t \equiv \ln P_t$, $\omega_t \equiv \ln w_t$ and σ is the standard deviation of innovations to the endowment process.

Since we derived equations (13) and (14) by plugging optimal allocations into the necessary and sufficient conditions (9), (10) for the economy's monetary equilibrium, equations (13) and (14) fully characterize the dynamics of the money stock and price level in the equilibrium that implements the optimal consumption allocation. In particular, from equations (13) and (14) it follows that the monetary equiblibrium and the optimal monetary policy are unique (up to a scaling factor for the money stock) for any sequence of endowments $\{\omega_t\}$. We summarize four main properties of price level dynamics under the optimal monetary policy:

(i) The average inflation under the optimal policy is positive, $\bar{\pi} = \frac{(1-\gamma)^2\sigma^2}{2}$, and increasing with the size of uncertainty, as long as $\gamma \neq 1$.[11]

(ii) Expected inflation is positively correlated with the current endowment:

$$E_t\left[p_{t+1} - p_t\right] = \bar{\pi} + \gamma(1-\rho)\omega_t. \tag{15}$$

(iii) The variance of expected inflation is decreasing in the persistence of the endowment process, ρ. If the endowment follows a random walk, $\rho = 1$, then optimal expected inflation is constant: $E_t\left[p_{t+1} - p_t\right] = \bar{\pi}$.

(iv) The log price level under the optimal policy is nonstationary.[12]

To understand these properties of the optimal policy, recall from equations (6) and (7) that the first-best allocation corresponds to the constant savings rate, $\frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}}$. In a monetary equilibrium, the savings rate depends on the expected return to money $E_t\left[p_t - p_{t+1}\right]$, which is the negative of the expected inflation. The monetary authority sets expected inflation, by appropriately choosing the rate of money growth, to stabilize the equilibrium savings rate at the optimal level. The first three properties of the optimal monetary policy describe how expected inflation must be set to achieve the first-best allocation. The last property, a nonstationary price level, is an outcome of the optimal monetary policy.

Property 1 is due to asset market incompleteness, implying that individuals cannot perfectly insure themselves against endowment risk. In the face of uncertainty about future income, risk-averse individuals have an incentive to self-insure by smoothing consumption

[11]This result requires that $E\left[\left(\frac{w_{t+1}}{w_t}\right)^{1-\gamma}\right] > 1$, and holds for any stationary distribution of endowment, and also for the case of random walk (with no drift).

[12]To show this we have to consider two cases: $\rho < 1$ and $\rho = 1$. First, assume $\rho < 1$. Suppose the money stock m_t is a stationary time series. Then it must be that $var(m_{t+1}) = var(m_t)$. Equation (13) then implies: $0 = (1-\rho)^2(\gamma-1)^2\frac{\sigma^2}{1-\rho^2} + 2cov(m_t, (1-\rho)(\gamma-1)\omega_t)$. Expanding m_t and ω_t, it is easy to see that the covariance term on the right-hand side is non-negative, leading to a contradiction. Thus, the money stock m_t cannot be stationary. Further, since $p_t = m_t - \log x_t$, and since $x_t \in [0, w_t]$ is stationary, the log of the price level must also be nonstationary.

If $\rho = 1$, then optimal x_t is nonstationary, while optimal m_t is stationary. This implies again that the log of the price level must be nonstationary.

across time. Without positive trend inflation, they tend to save more than optimal for precautionary reasons, as in Aiyagari (1994).[13] The positive average inflation serves as a tax on savings, which discourages oversaving.[14]

According to Property 2, a positive correlation between the expected inflation and income implies a high (low) expected tax on savings when income is high (low). This discourages individuals from varying savings rates to smooth consumption over time and thereby stabilizes the savings rate.

Property 3 implies that with higher endowment persistence individuals expect less variation in their marginal utility of consumption between their young and old ages. Therefore, they have less incentive to vary their savings rate. As a result, the optimal expected inflation has to vary less to discourage consumption smoothing. In the limit, when income follows a random walk, the optimal expected inflation is constant.

Finally, Property 4, a nonstationary price level, is the result of optimal expected inflation being independent of current and past price levels, i.e., optimal monetary policy being completely forward-looking. Under the optimal policy, it is expected inflation that matters for savings rates, and not the price level. Also note that without uncertainty, optimal expected inflation is always zero, and the optimal price level is constant. Hence, this paper emphasizes the role of monetary policy in affecting the savings behavior of risk-averse individuals under uncertainty and market incompleteness.

In Section 2.4, we demonstrate that the welfare ranking of other monetary policy regimes depends on how well they can approximate the four properties of optimal monetary policy. The persistence of the income shock is crucial in determining which properties are important.

2.4 Evaluating Targeting Regimes

In the previous section, we derived the unrestricted optimal monetary policy. In this section we shift attention from unrestricted optimal policies to the constrained monetary policy

[13]More precisely, Property 1 says that inflation under optimal policy under uncertainty is higher than without uncertainty (zero in this case). For example, for discounted social welfare function or in the infinite-horizon economy, optimal inflation without uncertainty is negative, so that with uncertainty, although higher, it can still be negative.

[14]This argument for positive inflation is different from the well known "dynamic inefficiency" result of OLG models. In our model, in the absence of shocks, the first-best allocation is attained with a constant money stock. Thus, our conclusions regarding the optimal monetary policy stem from the dynamic effect of incomplete markets in the presence of aggregate income uncertainty.

regimes.

Recently, both monetary policy theorists and practitioners have bcome interested in monetary policy regimes that constrain monetary policy to target a specific time path for a widely observable nominal variable (e.g., money growth rate, the exchange rate, or the inflation rate). In particular, central banks in a number of countries, such as Canada, Sweden, New Zealand, and the United Kingdom, adopted inflation targeting as their stated monetary policy regime. The focus of monetary-policy research has been to find the optimal targeting rule that the monetary authority should follow, or alternatively, to find the objective that should be "delegated" to the monetary authority.

In this section, we analyze a class of targeting rules that define monetary policy via its effect on the expected path of the price level: inflation targeting and price-level targeting. As equations (13) and (14) show, due to the timing restriction, monetary policy cannot fully control the price level, and so innovations to endowments act as shocks to the price level. Hence, both policy regimes target only the expected level of inflation or the price level, and allow deviations from the respective targets due to unexpected price shocks. In this subsection, we first introduce a parametrized class of constrained monetary policies that nest IT and PT. Next, we characterize the equilibrium in the overlapping-generations economy under the imposed constraint on monetary policy. Finally, we compare dynamics and welfare under IT and PT to those under the unrestricted optimal policy.

Suppose that the government sets the money growth rate so that in equilibrium the expected price level evolves according to the following dynamic equation

$$E_t\left[p_{t+1} - z_{t+1}\right] = \lambda(p_t - z_t) \tag{16}$$

where, $z_t = p_1 + \bar{\pi}t$, is a deterministic trend of the log price level and $\bar{\pi}$ is the average inflation rate. Since we focus on the dynamic properties of equilibria under policy rule (16), we assume that the average inflation rate is at its optimal level, $\bar{\pi} = \frac{(1-\gamma)^2\sigma^2}{2}$.

The adjustment coefficient, $\lambda \in [0, 1]$, determines how fast the price level is expected to return to the trend under targeting rule (16). In particular, $\lambda = 0$ corresponds to strict price level targeting, $\lambda = 1$ corresponds to strict inflation targeting, while any $\lambda \in (0, 1)$ correspond to gradual price level targeting where deviations from the trend are reversed more slowly than under strict price level targeting.[15] In general, the restricted set of simple

[15]We also consider an alternative class of monetary policy regimes: $E_t\left[p_{t+1} - p_t\right] = (1-\phi)\bar{\pi} + \phi\left(p_t - p_{t-1}\right),$

targeting rules (16) does not nest the (unrestricted) optimal monetary policy. Thus, the welfare attained under any monetary policy in the restricted class (16) will be less than that under the optimal monetary policy. We do not have a proof of the uniqueness of constrained policy rules. However, in computational experiments with different initial values, it appears that for each value of λ there is a unique monetary policy that satisfies (16).

Appendix C shows that targeting policy rules (16) are equivalent to a monetary authority choosing the price level path that minimizes the "loss function" delegated to it by the government. For example, a monetary authority that minimizes the loss function

$$L = E\sum_{t=1}^{\infty} \kappa^t (\pi_t - \overline{\pi})^2 \ ,$$

where $0 < \kappa < 1$, pursues the strict inflation targeting described by (16) with $\lambda = 1$. If the loss function is

$$L = E\sum_{t=1}^{\infty} \kappa^t (p_t - z_t)^2 \ ,$$

then the monetary authority follows the strict price-level targeting given by (16) with $\lambda = 0$.

In our model, we focus entirely on the saving decision and do not explicitly model labor-leisure or capital investment decisions. As a result, there is no inflation-output tradeoff, and thus no output gap term in any of the loss functions. Incorporating the labor-leisure and capital investment decisions in our framework is an interesting extension.

2.4.1 Characterizing Equilibria Under Targeting Policy Regimes

Define $x_t = \frac{M_t^d}{P_t}$, and $h_t = M_t \exp(-z_t)$. We can rewrite the first-order conditions (9) and (10) as

$$\exp(\overline{\pi}) x_t (w_t - x_t)^{-\gamma} = \beta \frac{h_t}{h_{t+1}} E_t \left[x_{t+1}^{1-\gamma} \right] \ ,$$

$$\frac{P_t}{P_{t+1}} = \frac{h_t}{h_{t+1}} \frac{x_{t+1}}{x_t} \exp(-\overline{\pi}) \ . \tag{17}$$

Appendix B shows that equation (17) together with policy rule (16) imply another dynamic equation

$$h_{t+1} = \left(\frac{h_t}{x_t} \right)^{\lambda} \exp(E_t \left[\ln x_{t+1} \right]). \tag{18}$$

where $\phi \in [0, 1]$. For $\phi > 0$, this regime corresponds to gradual inflation targeting, whereby deviations of inflation from $\overline{\pi}$ are gradually eliminated. It turns out that strict inflation targeting ($\phi = 0$) welfare dominates gradual inflation targeting ($\phi > 0$). Hence, we can focus our analysis on strict inflation targeting rules only.

Thus, to solve for a monetary equilibrium, we need to find sequences $\{x_t, h_t\}$ that satisfy the following two dynamic equations:

$$\exp(\bar{\pi}) \left[\frac{x_t}{h_t}\right]^{1-\lambda} (w_t - x_t)^{-\gamma} = \beta \exp(-E_t [\ln x_{t+1}]) E_t \left[x_{t+1}^{1-\gamma}\right] \tag{19}$$

$$h_{t+1} = \left(\frac{h_t}{x_t}\right)^{\lambda} \exp(E_t [\ln x_{t+1}]). \tag{20}$$

The state of the economy in period t can be fully summarized by $s_t = (\omega_t, h_t)$. To solve the system (19) and (20), we find functions $x_t = x(\omega_t, h_t)$ and $h_{t+1} = h(\omega_t, h_t)$ that satisfy the two dynamic equations.

2.4.2 Welfare Comparisons of Targeting Regimes

What constrained policy (parametrized by λ) in the class of targeting regimes (16) maximizes the unconditional expected utility attained in a stationary equilibrium? It is instructive to first consider the log utility case ($\gamma = 1$). For this special case it is easy to verify from equations (19) and (20) that the policy (16) that implements the first-best allocation corresponds to $\lambda = \rho$. Indeed, expected inflation under the optimal policy satisfies (15), which for this special case can be written as

$$E_t [p_{t+1} - p_t] = \gamma(1 - \lambda^*)\omega_t \ .$$

Hence, in the log utility case the targeting regime (16) with $\lambda^* = \rho$ attains the first-best allocation. It implies a constant money stock (see equation (13)), zero average inflation, and a stationary price level.

For $\gamma \neq 1$, there is also a special case for which an analytical solution exists. If the endowment process follows a random walk, $\rho = 1$, then the unrestricted optimal monetary policy is consistent with the restricted policy rule (16) at $\lambda^* = \rho = 1$. In this case, the expected inflation is constant

$$E_t [p_{t+1} - p_t] = \frac{(1-\gamma)^2\sigma^2}{2} = \bar{\pi} \ ,$$

and so inflation targeting attains the first-best allocation.

For other values of risk-aversion, γ, and income persistence, ρ, the welfare comparisons have to be done numerically. Welfare across equilibria are compared to the welfare in equilibrium under the optimal policy:

$$EV^* = \frac{1}{1-\gamma}\left(\frac{1}{1+\beta^{\frac{1}{\gamma}}}\right)^{-\gamma} \exp\left(\frac{(1-\gamma)^2\sigma^2}{2(1-\rho^2)}\right).$$

To calculate welfare under targeting rule (16), we simulate the economy over 10,000 periods, and calculate the average utility of all generations in the simulated sample, $V_T = \frac{1}{T}\sum_{t=1}^{T}[u(c_t^y) + \beta u(c_t^o)]$. We compare the value of the average utility, V_T, with EV^* by calculating the percentage by which consumption of every individual in every period has to be raised to reach the same level of welfare as under the optimal allocation. Specifically, let δ be a real number such that

$$\delta^{1-\gamma} V_T = \frac{1}{T}\sum_{t=1}^{T}\left[\frac{(\delta c_t^y)^{1-\gamma}}{1-\gamma} + \beta\frac{(\delta c_t^o)^{1-\gamma}}{1-\gamma}\right] = EV^*.$$

We report welfare loss as the net lifetime-consumption-equivalent compensation, $Comp.\% = (\delta - 1) * 100\%$.

Experimenting with various sets of parameters, we find that the discount rate, β, and the coefficient of relative risk aversion, γ, have little effect on the magnitude of welfare losses and on the relative welfare rankings. We pick standard values of these two parameters, $\beta = 0.96^{30}$ and $\gamma = 1.5$ (see Table 1). The value of the discount factor β ceteris paribus does not affect our results. In this sense the length of a period per ce does not affect our main conclusions. The standard deviation of innovations to the endowment, σ, affects the level, but not the order, of welfare losses under alternative targeting rules (16). Smaller values of σ imply smaller welfare differences between various policies. In particular, with no uncertainty, $\sigma = 0$, all policies (16) imply the same welfare loss. We set $\sigma = 0.08$.[16] As we noted earlier, income persistence, ρ, is crucial for welfare rankings of the alternative policy regimes, as well as for the magnitude of the welfare loss. For that reason, we report welfare losses for a range of values of income persistence.

Figure 1 presents the welfare losses across equilibria with various values of λ. We repeat the experiment for three different values of the serial correlation of the endowment process: $\rho = 0.1$, 0.5 and 0.9, keeping other parameters fixed. The welfare loss is U-shaped, with the minimum located close, but to the left of $\lambda = \rho$. At the minimum, the welfare loss is very

[16] Our estimates of σ from GDP data for U.S. and U.K. range from 0.02 to 0.16 depending on how we detrend the data, and what we assume about the stationarity of the income process. We choose a value in the middle of this range.

Table 1: Parameter values for model simulations

Parameter	Values
Discount factor, β	0.96^{30}
Relative risk aversion, γ	1.5
Standard deviation of innovations to endowment, σ	0.08
Persistence of the endowment process, ρ	0.1, 0.5, 0.9

small, less than 0.02% of consumption, so the best policy in the class of targeting regimes (16) is nearly optimal. For other policy rules in the restricted class, welfare losses grow faster as persistence of income decreases. In particular, this implies that under strict inflation targeting, welfare losses are disproportionately higher than under strict price-level targeting. For example, if $\rho = 0.1$, the welfare loss under the strict IT is 0.62%, whereas if $\rho = 0.9$, the welfare loss due to following the strict PT is much smaller, 0.08%.

We conclude that strict IT is closer to the (unrestricted) optimum if income fluctuations are highly persistent; and strict PT is closer to the optimum if income fluctuations are transitory. To understand the intuition behind this result, we examine how well each of the policy regimes replicates the four properties of the optimal monetary policy, discussed in Section 2.3.2.

First, we assume that IT and PT have the same trend inflation, $\overline{\pi}$, as the optimal policy, which allows us to focus on welfare differences due to the dynamic effects of monetary policy on savings under uncertainty. This assumption implies that without uncertainty, IT and PT would be equivalent not only to each other, but also to the optimal policy.

When aggregate uncertainty is present, income persistence is crucial for shaping the dynamic properties of monetary policy (Properties 2 and 3 in Section 2.3.2). If income follows a random walk, $\rho = 1$, optimal monetary policy requires constant expected inflation, which corresponds to inflation targeting. Thus, when the income process is highly persistent, strict IT is close to the optimum in terms of expected inflation dynamics, while strict PT implies too much variation in expected inflation. On the other hand, if income fluctuations are transitory, the optimal monetary policy requires a positive correlation between expected inflation and income, which is consistent with price-level targeting. Indeed, when income is high, a low realized price level calls for higher expected inflation to return the price level back to trend. Conversely, under inflation targeting, expected inflation is constant, hence its correlation with income is zero. Thus, when income persistence is low, strict PT is nearly optimal in terms of inflation dynamics, while strict IT implies too little variation in expected inflation.

The last property of the optimal policy is a nonstationary price level. Inflation targeting satisfies that property, while price-level targeting does not. Hence, optimal monetary policy combines elements of both price-level and inflation targeting: it reverts temporary deviations of the price level from the trend, but does not return the price level all the way back to the trend, as price-level targeting would prescribe. As we learned from model simulations, income persistence is key for the balance between these two features of the optimal policy. Depending on the persistence of income, the implied price level dynamics are mimicked better by either a strict price-level targeting or strict inflation targeting policy regime.

2.4.3 Optimal Horizon for Price-level Targeting

The model has simple policy implications for the targeting horizon: deviations of the price level from its target trend should be as persistent as income fluctuations. In other words, the best restricted policy regime corresponds to the transition coefficient, λ, being close to the persistence of endowment, ρ. Why is that so? As was demonstrated before, monetary policy provides high average welfare if it is able to stabilize the savings rate at the optimal level. In the model, the monetary authority controls the expected return on fiat money, $E_t\left[p_t - p_{t+1}\right]$, or equivalently, the expected inflation rate, $E_t\pi_{t+1} = E_t\left[p_{t+1} - p_t\right]$. So monetary policy yields high welfare if it is successful in replicating the path of expected inflation under the optimal policy :

$$E_t\left[\pi^*_{t+1}\right] = \overline{\pi} + \gamma(1-\rho)\omega_t \ . \tag{21}$$

In the special case of log utility, $\gamma = 1$, and $\lambda = \rho$, the equilibrium expected inflation is given by

$$E_t\left[\pi_{t+1}\right] = \overline{\pi} + \gamma(1-\lambda)\omega_t \ . \tag{22}$$

For risk aversion other than 1, if $\lambda \approx \rho$, the equilibrium expected inflation is approximately

$$E_t\left[\pi_{t+1}\right] \approx \overline{\pi} + \gamma(1-\lambda)\omega_t. \tag{23}$$

From (21) and (23) it follows that for λ close to ρ, the path of expected inflation under targeting rule (16) is going to be close to the optimal expected inflation.

2.4.4 What is Income Persistence?

Regrettably, there is no conclusive empirical evidence on the persistence of aggregate income. The empirical literature is mostly debating whether income is a trend-stationary variable or a nonstationary variable with drift. Campbell and Mankiw (1987) examine the persistence of U.S. real GNP by looking at the long-run impulse response in an estimated ARIMA model and conclude that shocks to GNP are largely permanent. However, they caution that it appears impossible to reject the view that output reverts to the trend after twenty years.

Cochrane (1988) questions those findings using a non-parametric variance test. He finds little persistence in real U.S. GNP, and asserts that conventional methods of estimating persistence are misleading, because they are trying to estimate a long-run impulse response from short-run dynamics. He concludes that the existence or size of a random walk component in GNP is not a precisely measured "stylized fact" that any reasonable model must reproduce, but warns that standard errors are large. Cochrane concludes that this result is unavoidable, since one needs long-run data to estimate long-run persistence of income, but there are inherently few nonoverlapping long runs available.

Perron (1989) also examines the persistence of U.S. GNP using the same dataset as Campbell and Mankiw (1987), and finds that traditional unit root tests cannot reject a unit root hypothesis if the true process is stationary around a trend which contains a one-time break. He develops a test which allows for a one time break in trend and finds that fluctuations are stationary around a deterministic trend function. The only shocks that had persistent effects are the 1929 crash and the 1973 oil price shock. Perron also cautions that the rejection of the unit root is conditional on treating one-time breaks in trend as exogenous events that are not part of the data-generating process. He further notes that even if these breaks are indeed part of the data-generating process, apparently they arrive extremely rarely.

Finally, Serletis (1992) replicates Perron's test using Canadian 1870-1985 real GNP and concludes that the unit root hypothesis could be rejected if allowance is made for the possibility of a one-time break of the trend function during the Great Depression.

Given these somewhat inconclusive results, we choose to report our findings for a range of values of the persistence parameter, ρ. The main finding of this section is theoretical. In a two-period overlapping-generations model with aggregate uncertainty, noncontingent nominal assets, and trend-stationary income, strict price-level targeting dominates strict inflation targeting and is close to the optimal policy in welfare terms. If income is nonstationary, then inflation targeting dominates price-level targeting and is close to the optimal policy. Given that the evidence of a unit root in aggregate income is rather weak, our results suggest that targeting rules that put significant weight on reverting temporary price-level deviations from the trend are likely to do better.

Overall, our results and intuition for optimal policy and targeting rules come from the analysis of savings decisions under uncertainty. Even though the benchmark model imposes strong assumptions about the structure of asset markets, the main findings can be generalized to more realistic environments. In particular, in the benchmark model there are no alternative assets except fiat money, and the nominal return on money is fixed (at unity). Under these assumptions, changes in expected inflation translate one for one into changes in real interest rates. If the economy had alternative assets in addition to money, with endogenously determined rates of return, then changes in inflation could have smaller effects on real interest rates. In Section 3 we present a richer model in which there are two assets besides money, and rates of return are determined endogenously. We find that our results for optimal policy and for targeting rules (16) apply with little change.

Later on in Section 4, we analyze a simple infinite-horizon economy with two types of agents who are heterogeneous in their endowment realizations. These agents face aggregate income risk and insure themselves imperfectly by trading an uncontingent nominal asset. We prove that the our results are valid in this infinite-horizon model and thus are not limited by the number of periods or the OLG structure.

3. An OLG Economy With Land, Nominal Bonds and Money

In order to keep things tractable, the model in the previous section has the strong assumption that there are no alternative assets, except fiat money. In this section, we present a richer model, in which there is productive land and nominal bonds in addition to money. Nominal bonds are different from money, and dominate it in the rate of return. We generate demand for money by assuming that young agents have to use money to buy land. The young borrow money from the old by issuing noncontingent nominal bonds. We assume that the trade of bonds and the purchase of land take place before the realization of the current productivity shock. One can think of these arrangements as mortgage contracts. The young take a mortgage, buy land, learn their productivity shock, produce, and then repay the bonds. The assumption of money being necessary to buy land, can be motivated by some underlying credit market intermediation technology, in which money serves as a medium of exchange. In this section, we first present the environment and characterize the monetary equilibrium. Then, we derive the optimal monetary policy which implements the first-best allocation, using the same welfare criterion as before. Finally, we will apply the model to compare welfare implications of PT and IT monetary policy regimes. Derivations will be relegated to the appendix. Despite the model being richer and more complex than the previous one, the results and intuition remain essentially unchanged.

3.1 The Environment

There is a unit measure of agents born every period. All individuals of the same generation are identical in all respects. Every generation lives for two periods. Further, each period is subdivided into two subperiods: morning and afternoon. In the morning of period t, the current productivity shock A_t has not been observed yet. The young generation has nothing except the endowment of labor. The old own the entire stock of land L plus the entire stock of money M_{t-1}. The government prints (destroys) new money in the amount $M_t - M_{t-1}$, and allocates it equally among the old with a lump-sum transfer (tax). The young borrow some amount of money from the old to finance their land purchases. The old lend money and then sell their land for money. The morning subperiod ends.

In the afternoon of period t the productivity shock A_t is realized. The young combine their labor endowment ($N = 1$) and the purchased land in a Cobb-Douglas production function to produce output, $Y_t = A_t L^\alpha N^{1-\alpha} = A_t L^\alpha$. They consume part of their output c_t^y, and use the remainder to retire their debt and to purchase money M_t^d for the next period, when they will be old. The afternoon subperiod ends.

We assume that the log of the productivity shock, $a_t = \ln A_t$, follows a first-order autoregressive process:

$$a_t = \rho a_{t-1} + \varepsilon_t \ ,$$

where ε_t is distributed as $N(0, \sigma^2)$. It is the only source of uncertainty in the model. In this section it is important to keep track of the timing of various events. For this reason we introduce state space notation and show the timing of events in the figure 3. We will use s^t to represent the history of the economy up to period t, and s_t to represent the current state of the economy. The history evolves as $s^{t+1} = (s^t, s_{t+1})$.Observe that with this notation, $M(s^{t-1})$ represents M_t, since we assume that M_t is determined in the morning of period t, that is, before $A_t = A(s^t)$ is realized.

The problem of the young is the following:

$$\max \sum \Pr(s^t|s^{t-1})u(c^y(s^t)) + \beta \sum \Pr(s^{t+1}|s^t)u(c^o(s^{t+1}))$$

subject to the following constraints:

- In the morning of period t, before A_t is realized

$$q(s^{t-1})L^d(s^{t-1}) \leq d(s^{t-1})B^d(s^{t-1}) \ ,$$

  where $q(s^{t-1})$ is the morning nominal price of land, $d(s^{t-1})$ is the discount factor on money borrowed in the morning, and $B^d(s^{t-1})$ is the nominal value of goods to be returned to the old in the afternoon for repayment of debt.

- In the afternoon of period t, after A_t is realized

$$c^y(s^t) + \frac{M^d(s^t)}{P(s^t)} \leq A_t \left(L^d(s^{t-1})\right)^\alpha - \frac{B^d(s^{t-1})}{P(s^t)} \ .$$

- In the morning of period $t+1$, before A_{t+1} is realized

$$d(s^t)B^s(s^t) \leq M^d(s^t) + T(s^t) \ .$$

- In the afternoon of period $t+1$, after A_{t+1} is realized

$$c^o(s^{t+1}) \leq \frac{B^s(s^t)}{P(s^{t+1})} + \frac{q(s^t)L^s(s^t) + \left(M^d(s^t) + T(s^t) - d(s^t)B^s(s^t)\right)}{P(s^{t+1})} \ ,$$

  where $T(s^t) = M(s^t) - M(s^{t-1})$.

In equilibrium it must be the case that

$$\begin{aligned} L^s(s^t) &= L^d(s^{t-1}) = L^d(s^t) = L \ , \\ B^s(s^t) &= B^d(s^t). \end{aligned}$$

It remains to specify a monetary policy to complete the description of the model. Again, we will start from the optimal monetary policy.

3.2 Optimal Monetary Policy

We maintain the same social welfare criterion as in Section 2.3.1. In Appendix E we solve the social planner's problem for the first-best allocations and then derive the optimal monetary policy that implements it. The optimal monetary policy implies the following price-level dynamics:

$$p(s^{t+1}) - p(s^t) = \ln\left(\frac{1}{d(s^t)}\right) + \frac{(1-\gamma)^2\sigma^2}{2} + \gamma(1-\rho)a_t - \varepsilon_{t+1}.$$

Observe that, except for the first term on the right hand side, this is exactly the same formula as in the money only economy. The discount rate $d(s^t)$ is a constant in three special cases: first, when shocks to productivity are i.i.d. ($\rho = 0$), second, when shocks to productivity follow a random walk ($\rho = 1$), and third, when utility is logarithmic ($\gamma = 1$). Outside of these special cases, the discount rate does vary with productivity, but fluctuations are small. Thus, we can state essentially the same results as in the money only economy:

(i) Average inflation under the optimal policy is positive, $\overline{\pi} = E\left[\ln\left(\frac{1}{d(s^t)}\right)\right] + \frac{(1-\gamma)^2\sigma^2}{2}$. [17]

(ii) Expected inflation is positively correlated with current income

$$E_t\left[p_{t+1} - p_t\right] = \ln\left(\frac{1}{d(s^t)}\right) + \frac{(1-\gamma)^2\sigma^2}{2} + \gamma(1-\rho)a_t.$$

(iii) The variability of expected inflation is decreasing in persistence of the endowment process, ρ. In particular, when $\rho = 1$, the optimal expected inflation is constant: $E_t\left[p_{t+1} - p_t\right] = \overline{\pi}$.

(iv) The optimal price level is nonstationary.

Let us now see if our results for PT/IT monetary policy regimes also carry over to this model.

3.3 Inflation Targeting and Price Level Targeting Policy Regimes

As in the money only economy, we apply the model to compare welfare in monetary equilibria with inflation and price level targeting policy rules. As before, we refer to these constrained monetary policies as targeting rules. In Appendix D we show how to solve the model and outline the computation procedure.

[17]Note that that $\overline{\pi}$ is not the same as before, but it is still the optimal long-run inflation.

What constant value of λ maximizes the unconditional expected utility $E\left[u(c^{y}(s))+\beta u(c^{o}(s'))\right]$, attained in a monetary equilibrium? There are again the same special cases, for which we can solve the model analytically. We first look at the log utility case ($\gamma=1$). For this special case it is easy to (guess and) verify from the first-order optimality conditions (see equations (D10) - (D12) in Appendix D) that the optimal policy implementing the first-best allocation is consistent with (16) if $\lambda=\rho$.

Similarly, if the endowment process is a random walk, $\rho=1$, then (16) with $\lambda=\rho=1$ is consistent with the optimal policy implementing the first-best allocations.

For other values of parameters we have to resort to simulations. We also need to choose parameters. The model is homogeneous in L, so without loss of generality, we can normalize it to unity[18]. The advantage of doing this is that income is now equal to productivity A_t. As a result we can use the same parameters for the stochastic process of productivity, as the ones for the endowment in the money only economy. Therefore, we use the same set of parameters as before, except for one additional parameter, the income share of land. The income share of land, α, in the production function is taken as the share of structures in GDP reported in Krussel et al. (2000). Table 2 lists all the parameters.

Table 2: Parameter values for model simulations, land economy

Parameter	Values
Discount factor, β	0.96^{30}
Relative risk aversion, γ	1.5
Standard deviation of innovations to endowment, σ	0.08
Persistence of endowment process, ρ	0.1, 0.5, 0.9
Income share of land, α	0.12

We simulate the economy with different values of λ over many periods, and for each rule we calculate the average utility of all generations in the simulated sample, $V_T=\frac{1}{T}\sum_{t=1}^{T}\left[u(c_t^y)+\beta u(c_t^o)\right]$. We compare the value of the average utility, V_T, with the maximum expected utility, EV^*, computed at the first-best allocations:

$$c_t^{y*} = \frac{1}{1+\beta^{\frac{1}{\gamma}}}A_tL^{\alpha},$$
$$c_t^{o*} = \frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}}A_tL^{\alpha}.$$

[18] Consumption equivalent welfare losses are independent of L.

With the optimal allocation, and $L = 1$ the unconditional expected utility is

$$EV^* = \frac{1}{1-\gamma}\left(\frac{1}{1+\beta^{\frac{1}{\gamma}}}\right)^{-\gamma} \exp\left(\frac{(1-\gamma)^2\sigma^2}{2(1-\rho^2)}\right) ,$$

as in the money only model. As a welfare metric, we use the same consumption equivalent compensation measure as before. Results of these simulations are presented in Figure 2 which presents welfare losses across equilibria with various values of λ. As is evident from Figure 2, results are very similar to those from the money only economy: income persistence determines whether PT or IT is closer to the optimum, and the best targeting regime has $\lambda \approx \rho$. Thus our results transfer with almost no change to this richer environment with additional assets.

4. An infinite-horizon Economy

Next we show that our results are not limited to the overlapping generations environment. To do this, we analyze a simple model economy in which instead of the overlapping generations structure we assume that individuals are infinitely lived. We retain all the other main ingredients of the previous models: aggregate-endowment uncertainty, ex-post endowment heterogeneity among individuals, and market incompleteness due to uncontingent nominal assets being the only means of savings.

4.1 The Environment

There are two types of individuals: measure one of the “Odd” type agents who receive all of the aggregate endowment w_t in the odd periods $t = 1, 3, 5...$ and measure one of the “Even” type of agents who receive all of the aggregate endowment w_t in the even periods $t = 2, 4, 6....$ As before, the log of the aggregate endowment follows a first-order autoregressive process from one period to the next :

$$\ln w_t = \rho \ln w_{t-1} + \varepsilon_t.$$

The single asset in the economy is fiat money supplied by the government. In period 1 the “Even” agents that have no endowment in that period, hold M_0 units of the money stock.[19]

The timing of events is as follows: at the beginning of period t the agents who do not

[19]The assumption of zero endowment realizations in some periods is without loss of generality. Existence of monetary equilibrium requires that ”Odd” agents have more endowment than ”Even” agents in the odd periods (and less in the even periods) and ”Even” agents holding assets in period 1.

receive the endowment in that period, hold M_{t-1} units of fiat money which they acquired in the previous period. Before the current endowment w_t is realized, the government prints (or destroys) money in the amount of $M_t - M_{t-1}$, and distributes it evenly among the individuals from the money holding group via lump-sum transfer (or tax if negative) $T_t = M_t - M_{t-1}$. The assumption that monetary transfers occur before the realization of the current endowment, once again, implies an incomplete degree of control over the price level. After the realization of the current endowment, w_t, the agents of type i $(i = e, o)$ who receive it in period t, consume c_t^i units of it. The remaining goods, $w_t - c_t^i$, are exchanged for M_t^d units of money at the price P_t. The money is supplied by the agents from the other group j $(j = o, e)$, who have M_t and consume $c_t^j = \frac{M_t^d}{P_t} \leq \frac{M_t}{P_t}$.

Thus, an "Odd" person in this economy solves the following problem:

$$\max E_1 \sum_{t=1}^{\infty} \beta^{t-1} u(c_t^o)$$

subject to

$$\begin{aligned} P_t c_t^o + M_t^d &\leq P_t w_t \ , \text{ in } t = 1, 3, 5, ... \\ P_t c_t^o &\leq M_{t-1}^d + T_t \ , \text{ in } t = 2, 4, 6, ..., \end{aligned} \tag{24}$$

while an "Even" person solves:

$$\max E_1 \sum_{t=1}^{\infty} \beta^{t-1} u(c_t^e)$$

subject to

$$\begin{aligned} P_t c_t^e + M_t^d &\leq P_t w_t \ , \text{ in } t = 2, 4, 6, ... \\ P_t c_t^e &\leq M_{t-1}^d + T_t \ , \text{ in } t = 1, 3, 5, 7... \ . \end{aligned} \tag{25}$$

The goods and money market clearing conditions are the same as before:

$$\begin{aligned} c_t^e + c_t^o &= w_t \ , \\ M_t^d &= M_t \end{aligned}$$

in all periods.

4.2 The Social Planner's Problem

As before, we start by defining the social welfare function and solving the social planner's problem for the optimal allocation. We then ask whether this allocation can be implemented as a monetary equilibrium.

For the sake of simplicity, we abstract from lifetime wealth differences among agents induced by the different timing of their endowment realizations. Instead we assume that in period 0 all agents have equal probability of becoming "Odd" or "Even" in period 1. So, in period zero, all agents are ex-ante identical, and in the presense of complete markets would have the same life-time consumption and wealth. The social planner's problem that implements such an ex-ante first-best allocation assigns the same welfare weight to agents of both types. We can state the Social Planner's problem as follows:

$$\begin{aligned} V_T &= \frac{1}{2}\left\{\sum_{t=1}^{\infty}\beta^t u(c_t^o) + \sum_{t=1}^{\infty}\beta^t u(c_t^e)\right\} \\ &= \frac{1}{2}\sum_{t=1}^{\infty}\beta^t\left[u(c_t^o) + u(c_t^e)\right]. \end{aligned}$$

subject to the resource constraint for all periods:

$$c_t^o + c_t^e \leq w_t, \text{ for } t = 1, 2, \dots .$$

It is easy to see that the solution to this problem is the sequence of consumptions $\{c_t^y, c_t^o\}_{t=1}^{\infty}$ such that in each period

$$c_t^e = c_t^o = \frac{1}{2}w_t \ . \tag{26}$$

4.3 Monetary Equilibrium

Any monetary equilibrium must satisfy the first-order conditions:

$$u'(w_t - \frac{M_t^d}{P_t}) = \beta E_t\left[u'(\frac{M_t^d + T_{t+1}}{P_{t+1}})\frac{P_t}{P_{t+1}}\right], \tag{27}$$

$$\frac{P_t}{P_{t+1}} = \frac{M_t}{M_{t+1}}\frac{w_{t+1} - c_{t+1}^i}{w_t - c_t^j}, \tag{28}$$

and the standard transversality condition. Here $(i,j) = (o,e)$, or $(i,j) = (e,o)$ depending on the period. Equation (27) is a standard intertemporal Euler condition. Equation (28) is derived from the budget constraint (24) respectively (25) holding with equality, and from the money market clearing condition $M_t^d = M_t$.

Clearly equilibria in the infinite-horizon economy characterized by (27)-(28) and in the OLG economy characterized by (9)-(10) are equivalent up to renaming the agents. Consequently, all qualitative results established in Section 2 remain unchanged.[20] Quantitative results change only to the extent that the calibrated values of the income persistence ρ and of the standard deviation of income fluctutaions σ would have to be adjusted to a shorter period length. A larger value of ρ makes the optimal persistence of price fluctuations, λ, higher, while smaller σ scales down the welfare differences between alternative policies.

To sort the results of this section into the overall picture: We have shown that neither the assumption of an OLG structure nor that of a two perod life span are essential for our main results. Indeed it appears that the basic features of the optimal monetary policy will extend beyond our basic model. Key ingredients to this result are aggregate uncertainty, incomplete markets and ex-post income heterogeneity across agents and time. These elements make nominal uncontingent assets essential and create demand for trade that facilitates risk sharing.

5. Conclusion

We explore the role of monetary policy in the environment with aggregate risk, incomplete markets and long-term nominal bonds. In a two-period overlapping-generations model with aggregate uncertainty and nominal bonds, optimal monetary policy attains the ex-ante Pareto optimal allocation. This policy is characterized by a premium on the average inflation, a positive correlation between expected inflation and income, an inverse relationship between volatility of expected inflation and persistence of income, and a nonstationary price level.

[20]The only difference is that the optimal average inflation is $\bar{\pi} = \frac{(1-\gamma)^2\sigma^2}{2}$ in the OLG model and $\bar{\pi} = \ln\beta + \frac{(1-\gamma)^2\sigma^2}{2}$ in the infinite-horizon model. Thus, for β sufficiently lower than one, the optimal average inflation rate is negative in the infinite horizon economy, while it is positive in the OLG model. If we used a discounted social welfare function (8) in our OLG model, we would obtain exactly the same optimal average inflation as in the infinite horizon model. As we discussed before, however, the first-best allocation that one obtains with the discounted welfare function does not maximize ex-ante utility of individuals in the OLG economy, because they discount old-age consumption and would prefer a path of consumption that gives higher expected consumption to the young agents. The fact that the infinite horizon model looks very much like an OLG model with the discounted welfare function makes us question whether an infinite horizon model is a good approximation to an OLG economy with altruistic links between generations.

The results extend to more general environments with multiple assets and infinitely-lived agents. The model sheds light on the debate over the advantages of price level targeting regimes relative to inflation targeting regimes. Persistence of income is key. If the income process is a random walk, inflation targeting is the optimal monetary policy, while price-level targeting is not. When income persistence is low price-level targeting dominates inflation targeting and is close to the first best monetary policy in welfare terms.

References

Aiyagari, S.R. 1994. "Uninsured Idiosyncratic Risk and Aggregate Saving." *The Quarterly Journal of Economics* 109(3): 659–84.

Akyol, A. 2004. "Optimal monetary policy in an economy with incomplete markets and idiosyncratic risk." *Journal of Monetary Economics* 51(6): 1245–1269.

Ball, L. and N. Sheridan. 2003. "Does Inflation Targeting Matter?" NBER Working Paper 9577.

Campbell, J. and G. Mankiw. 1987. "Permanent and Transitory Components in Macroeconomic Fluctuations." *American Economic Review* 72(2): 111–17.

Carlstrom, C.T. and T.S. Fuerst. 2002. "Monetary Policy Rules and Stability: Inflation Targeting versus Price-Level Targeting." Federal Reserve Bank of Cleveland, Economic Commentary (April).

Champ, B. and S. Freeman. 2001. *Modeling Monetary Economies.* Cambridge University Press.

Cochrane, J. 1988. "How Big Is the Random Walk in GNP." *The Journal of Political Economy* 96(5): 893–920.

Dittmar, R. and W.T. Gavin. 2005. "Inflation-Targeting, Price-Path Targeting and Indeterminacy." *Economic Letters* 88(3): 336–42.

Dittmar, R., W.T. Gavin, and F.E. Kydland. 1999. "Price-Level Uncertainty and Inflation Targeting." Review, Federal Reserve Bank of St. Louis, July Issue 23–34.

Duguay, P. 1994. "Some Thoughts On Price Stability Versus Zero Inflation." Mimeo, Bank of Canada .

Gavin, W.T. and A.C. Stockman. 1991. "Why a Rule For Stable Prices May Dominate a Rule for Zero Inflation." *Economic Review,* Q I, 2–8.

Krussel, P., L.E. Ohanian, J.V. Rios-Rull, and G.L. Violante. 2000. "Capital Skill Complementarity and Inequality: A Macroeconomic Analysis." *Econometrica* 68(5): 1029–54.

Ortega, E. and N. Rebei. 2006. "The Welfare Implications of Inflation versus Price-Level Targeting in a Two-Sector, Small Open Economy." Bank of Canada Working Paper 2006-12 .

Perron, P. 1989. "The Great Crash, the Oil Price Shock, and the Unit Root Hypothesis." *Econometrica* 57(6): 1361–1401.

Serletis, A. 1992. "The Random Walk in Canadian Output." *Canadian Journal of Economics* 25(2): 392–406.

Smets, F. 2003. "Maintaining Price Stability: How Long is the Medium Run?" *Journal of Monetary Economics* 50(6): 1293–1309.

Svensson, L. 1999. "Price-Level Targeting versus Inflation Targeting: A Free Lunch?" *Journal of Money, Credit and Banking* 31(3): 277–295.

Vestin, D. 2000. "Price-Level Targeting Versus Inflation Targeting in a Forward-Looking Model." Working Paper Series ISSN 1402-1903, No. 106, Sveriges Riksbank .

Wallace, N. 1992. "The Overlapping Generations Model of Fiat Money." *The New Classical Macroeconomics* 2: 379–412.

Walsh, C.E. 1998. *Monetary Theory and Policy.* The MIT Press.

Woodford, M. 2003. *Interst and Prices, Foundations of a Theory of Monetary Policy.* Princeton University Press.

Yetman, J. 2005. "The Credibility of the Monetary Policy: 'Free Lunch'." *Journal of Macroeconomics* 27(3): 434–51.

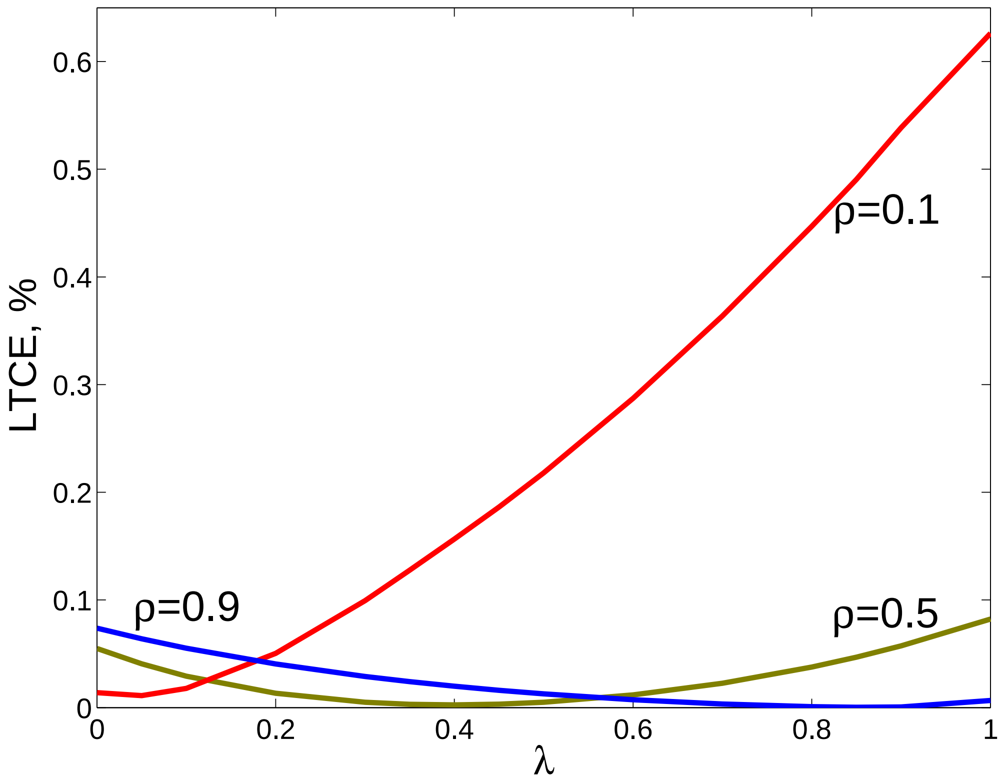

Figure 1: Welfare losses implied by constrained policy rules for three different degrees of endowment persistence. Benchmark OLG model with money only. Parameters: $\beta = 0.96^{30}, \lambda = 1.5, \sigma = 0.08$.

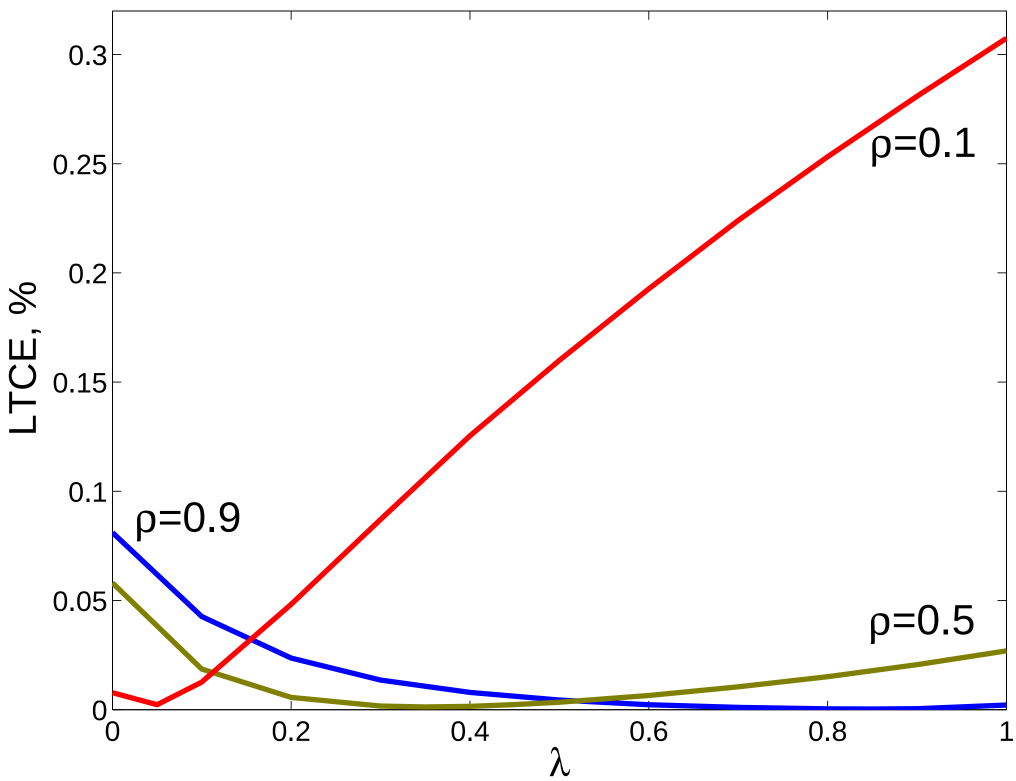

Figure 2: Welfare losses implied by constrained policy rules for three different degrees of endowment persistence. OLG model with money, land and nominal bonds. Parameters: $\beta = 0.96^{30}, \lambda = 1.5, \sigma = 0.08, \alpha = 0.12$.

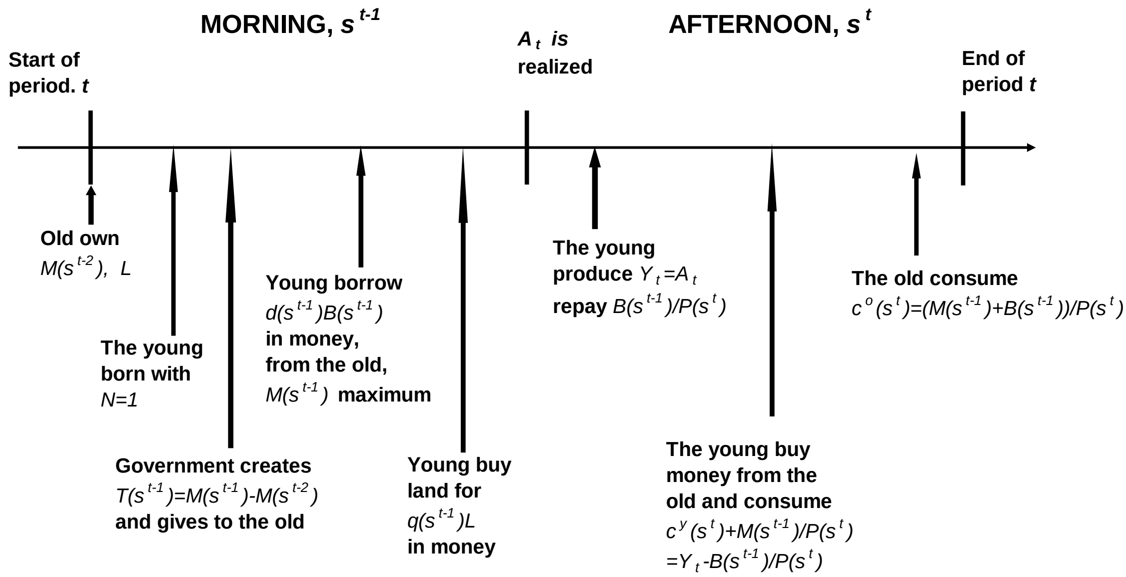

Figure 3: Timing of events in the model with land

Appendix A: The Solution To the Social Planner's Problem

Suppose, for a given history of endowment realizations, $w^T = \{w_1, w_2, ..., w_T\}$, we are solving the following problem:

$$\max V_T$$

$$\begin{aligned} \text{subject to} &: \\ c_t^y + c_t^o &\leq w_t, \text{ for all } t = 1, 2, ..., T. \end{aligned} \tag{A1}$$

The solution of this problem is $\{c_t^y, c_t^o\}_{t=1}^T$ such that:

$$\begin{aligned} u'(c_t^y) &= \beta u'(c_t^o), \\ c_t^y + c_t^o &= w_t. \end{aligned}$$

It is a pair of consumption functions $c_t^y = c^{y*}(w_t)$, and $c_t^o = c^{o*}(w_t)$. Given w_t, they are independent of T and of the realized endowment history w^T.

Let

$$V_T^* = \frac{1}{T} \sum_{t=1}^{T} \left[u(c^{y*}(w_t)) + \beta u(c^{o*}(w_t)) \right].$$

Let $\{c_t^y, c_t^o\}_{t=1}^T$ be any other sequence of consumptions that satisfies (A1) in each period t, and let V_T be the corresponding average ex-post utility as defined in (4). Then $V_T^* \geq V_T$, since for all $t = 1, 2..., T$ we have

$$u(c^{y*}(w_t)) + \beta u(c^{o*}(w_t)) \geq u(c_t^y) + \beta u(c_t^o). \tag{A2}$$

Taking expectation of $V_T^* - V_T$ with respect to realizations of w^T we have:

$$E[V_T^*] - E[V_T] \geq 0.$$

Taking the liminf with respect to T we have

$$\lim_{T \to \infty} \inf \left(E[V_T^*] - E[V_T] \right) \geq 0.$$

Since the sequence $\{c_t^y, c_t^o\}_{t=1}^T$ was arbitrary, the stationary policy $c^{y*}(w), c^{o*}(w)$ attains the maximum of the expected average utility, $E[V_T]$, for all T.

Appendix B: Derivation of Equation (18)

Let $x_t = \frac{M_t}{P_t}$ be the real money balances in period t. With this notation, the real return on money is

$$\frac{P_t}{P_{t+1}} = \frac{M_t}{M_{t+1}} \frac{x_{t+1}}{x_t}. \qquad \text{(B1)}$$

Let us rewrite equation (B1) in log terms

$$p_t - p_{t+1} = m_t - m_{t+1} + \ln x_{t+1} - \ln x_t. \qquad \text{(B2)}$$

We concentrate on $PT(\lambda)$ monetary policy rules that satisfy the dynamic equation

$$p_t - z_t = \lambda(p_{t-1} - z_{t-1}) - v_t, \qquad \text{(B3)}$$

where v_t is a zero-mean, serially-uncorrelated stochastic process, that is orthogonal to the information set available at the end of period t.

We first need to find the money supply rule that will be consistent with (B3). Equation (B3) implies

$$p_t - p_{t+1} = \lambda(p_{t-1} - p_t) - (1 - \lambda)\bar{\pi} - v_t + v_{t+1}. \qquad \text{(B4)}$$

Let $\xi_t = \ln x_t$. From equations (B3) and (B4) it follows that the monetary policy rule that is consistent with (16) must satisfy:

$$m_{t+1} - m_t = \lambda(p_t - p_{t-1}) + (1 - \lambda)\bar{\pi} + \xi_{t+1} - \xi_t + v_t - v_{t+1}.$$

Taking conditional expectation E_t on both sides, we obtain the money supply rule:

$$m_{t+1} - m_t = \lambda(p_t - p_{t-1}) + (1 - \lambda)\bar{\pi} + E_t\left[\xi_{t+1}\right] - \xi_t + v_t, \qquad \text{(B5)}$$

which implies that $v_{t+1} = \xi_{t+1} - E_t\left[\xi_{t+1}\right]$, and $E_t\left[v_{t+1}\right] = 0$. Thus, v_{t+1} is indeed an innovation, and is, therefore, orthogonal to all the variables known as of period t:

$$E_t\left[\Omega_t v_{t+1}\right] = \Omega_t E_t\left[v_{t+1}\right] = 0.$$

Further, since $E(v_t v_{t+1}) = E\left[E_t(v_t v_{t+1})\right]$ by the law of iterated expectations, and since $E_t(v_t v_{t+1}) = v_t E_t(v_{t+1}) = 0$, it follows that $E(v_t v_{t+1}) = 0$. Therefore, the error process v_t is serially uncorrelated. Thus, the government must change the money supply according to (B5) to guarantee that the price level follows the dynamic equation (B3) in each period.

We are ready to derive equation (18). From (B3) and the fact that $P_t = \frac{M_t}{x_t}$ it follows that

$$\frac{M_{t+1}}{x_{t+1}Z_{t+1}} = \left(\frac{M_t}{x_t Z_t}\right)^{\lambda} \exp(-v_{t+1}).$$

Which implies

$$\begin{aligned}
\frac{M_{t+1}}{Z_{t+1}} &= \left(\frac{M_t}{Z_t}\right)^{\lambda} \frac{x_{t+1}}{x_t^{\lambda}} \exp(-\xi_{t+1} + E_t\left[\xi_{t+1}\right]) \\
&= \left(\frac{M_t}{Z_t}\right)^{\lambda} \frac{x_{t+1}}{x_t^{\lambda}} \exp(-\ln x_{t+1}) \exp(E_t\left[\ln x_{t+1}\right]) \\
&= \left(\frac{M_t}{Z_t}\right)^{\lambda} \frac{\exp(E_t\left[\ln x_{t+1}\right])}{x_t^{\lambda}}.
\end{aligned}$$

Denoting $h_t = \frac{M_t}{Z_t}$ we obtain the desired equation (18)

$$h_{t+1} = \left(\frac{h_t}{x_t}\right)^{\lambda} \exp(E_t\left[\ln x_{t+1}\right]).$$

Appendix C: Equivalence of Loss Functions and Strict Targeting Rules

Suppose the monetary authority minimizes the loss function

$$L = E \sum_{t=1}^{\infty} \kappa^t (\pi_t - \overline{\pi})^2, \tag{C1}$$

where κ is an arbitrary constant in $(0,1)$, subject to $\{\pi_t\}_{t=1}^{\infty}$ being consistent with a monetary equilibrium. The constraint says that in choosing its monetary policy, the monetary authority must to respect the first-order conditions of individuals. Ignoring the constraint for a moment, the solution of an unconstrained problem is

$$\min L = E \sum_{t=1}^{\infty} \kappa^t (\pi_t - \overline{\pi})^2.$$

Due to our timing assumption, the monetary authority has control of next period's expected inflation only, and is not able to control next period's realized inflation. Thus, we can rewrite the problem as

$$\min L = E \sum_{t=1}^{\infty} \kappa^t (E_{t-1}\pi_t + v_t - \overline{\pi})^2, \tag{C2}$$

where v_t is the part of inflation in period t that is not under the control of the monetary authority. The first order condition of the unconstrained problem is

$$2\kappa^t E_{t-1}(E_{t-1}\pi_t + v_t - \overline{\pi}) = 0,$$

or equivalently,

$$E_{t-1}\left(\pi_t - \overline{\pi}\right) = 0.$$

If the monetary authority sets $E_{t-1}\pi_t = \overline{\pi}$ every period, then the first order conditions of the unconstrained problem are satisfied. Under strict IT, the monetary authority is required to have $E_{t-1}\pi_t = \overline{\pi}$. In any monetary equilibrium with this constraint imposed, the first order conditions of individuals hold as well. It follows then, that any strict IT regime monetary equilibrium, satisfies the necessary and sufficient conditions of the constrained problem with the loss function given by (C1).

To show the converse, suppose there is a solution of the constrained problem with the loss function (C1) such that $E_{t-1}\pi_t \neq \overline{\pi}$ in at least one period with positive probability. This means that the constraint set must be binding, and the value of the loss function must be greater relative to the the unconstrained problem. This leads to a contradiction, because any strict IT regime equilibrium minimizes the unconstrained loss function, and thus must do better than our assumed solution. This proves the equivalence.

For the strict PT regime and the loss function $L = E\sum_{t=1}^{\infty}\kappa^t(p_t - z_t)^2$, the logic is identical.

Appendix D: Solving the Model With Land

We can state the problem of the young born in period t as follows:

$$\max \sum \Pr(s^t|s^{t-1}) u\left(A_t\left(L^d(s^{t-1})\right)^\alpha - \frac{q(s^{t-1})L^d(s^{t-1})}{d(s^{t-1})P(s^t)} - \frac{M^d(s^t)}{P(s^t)}\right)$$
$$+\beta \sum \Pr(s^{t+1}|s^t) u\left(\frac{B^s(s^t)}{P(s^{t+1})} + \frac{q(s^t)L^d(s^{t-1}) + \left(M^d(s^t)+T(s^t)-d(s^t)B^s(s^t)\right)}{P(s^{t+1})}\right),$$

subject to the constraint

$$d(s^t)B^s(s^t) \leq M^d(s^t) + T(s^t). \tag{D1}$$

The first-order conditions of the problem are:

$$0 = \sum \Pr(s^t|s^{t-1}) \left[\begin{array}{c} u'\left(A_t L^\alpha - \frac{q(s^{t-1})L}{d(s^{t-1})P(s^t)} - \frac{M^d(s^t)}{P(s^t)}\right)\left(\alpha A_t L^{\alpha-1} - \frac{q(s^{t-1})}{d(s^{t-1})P(s^t)}\right) + \\ \beta \sum \Pr(s^{t+1}|s^t)\left\{u'\left(\frac{B^s(s^t)}{P(s^{t+1})} + \frac{q(s^t)L + \left(M^d(s^t)+T(s^t)-d(s^t)B^s(s^t)\right)}{P(s^{t+1})}\right)\frac{q(s^t)}{P(s^{t+1})}\right\}\end{array}\right], \tag{D2}$$

$$\begin{aligned} 0 \;=\;& \xi(s^t) - u'\left(A_t L^\alpha - \frac{q(s^{t-1})L}{d(s^{t-1})P(s^t)} - \frac{M^d(s^t)}{P(s^t)}\right)\frac{1}{P(s^t)} + \\ & \beta \sum \Pr(s^{t+1}|s^t)\left[u'\left(\frac{B^s(s^t)}{P(s^{t+1})} + \frac{q(s^t)L + \left(M^d(s^t)+T(s^t)-d(s^t)B^s(s^t)\right)}{P(s^{t+1})}\right)\frac{1}{P(s^{t+1})}\right], \end{aligned} \tag{D3}$$

$$\begin{aligned} & \beta \sum \Pr(s^{t+1}|s^t)\left[u'\left(\frac{B^s(s^t)}{P(s^{t+1})} + \frac{q(s^t)L + \left(M^d(s^t)+T(s^t)-d(s^t)B^s(s^t)\right)}{P(s^{t+1})}\right)\frac{1-d(s^t)}{P(s^{t+1})}\right] \\ =\;& \xi(s^t)d(s^t), \end{aligned} \tag{D4}$$

where $\xi(s^t) \geq 0$ is the Lagrange multiplier on constraint (D1).

From the last first-order condition it follows that $d(s^t) \leq 1$. It must be strictly less than 1 in equilibrium, and this implies that old agents will lend all of their money stock, while young agents will borrow just enough to buy land. So, in equilibrium we must have

$$
\begin{aligned}
M^d(s^t) &= M(s^{t-1}), \\
q(s^t)L &= d(s^t)B^s(s^t) = M(s^{t-1}) + T(s^t) = M(s^t).
\end{aligned}
\tag{D5}
$$

Taking into account the equilibrium conditions (D5), we can rewrite the first order conditions (D2)-(D4) as follows

$$
0 = \sum \Pr(s^t|s^{t-1}) \left[\begin{array}{c} u'\left(A_t L^\alpha - \frac{M(s^{t-1})}{d(s^{t-1})P(s^t)} - \frac{M(s^{t-1})}{P(s^t)}\right)\left(\alpha A_t L^{\alpha-1} - \frac{M(s^{t-1})/L}{d(s^{t-1})P(s^t)}\right) \\ +\beta \sum \Pr(s^{t+1}|s^t)\left\{u'\left(\frac{M(s^t)}{d(s^t)P(s^{t+1})} + \frac{M(s^t)}{P(s^{t+1})}\right)\frac{M(s^t)/L}{P(s^{t+1})}\right\} \end{array} \right],
$$

$$
\begin{aligned}
0 &= \xi(s^t) - u'\left(A_t L^\alpha - \frac{M(s^{t-1})}{d(s^{t-1})P(s^t)} - \frac{M(s^{t-1})}{P(s^t)}\right)\frac{1}{P(s^t)} \\
&\quad +\beta \sum \Pr(s^{t+1}|s^t)\left[u'\left(\frac{M(s^t)}{d(s^t)P(s^{t+1})} + \frac{M(s^t)}{P(s^{t+1})}\right)\frac{1}{P(s^{t+1})}\right],
\end{aligned}
$$

$$
\xi(s^t)d(s^t) = \beta \sum \Pr(s^{t+1}|s^t)\left[u'\left(\frac{M(s^t)}{d(s^t)P(s^{t+1})} + \frac{M(s^t)}{P(s^{t+1})}\right)\frac{1-d(s^t)}{P(s^{t+1})}\right].
$$

From the last equation we can find $\xi(s^t)$

$$
\xi(s^t) = \frac{1-d(s^t)}{d(s^t)}\beta \sum \Pr(s^{t+1}|s^t)\left[u'\left(\frac{M(s^t)}{d(s^t)P(s^{t+1})} + \frac{M(s^t)}{P(s^{t+1})}\right)\frac{1}{P(s^{t+1})}\right], \tag{D6}
$$

and then substitute the expression for $\xi(s^t)$ into the previous two equations

$$
0 = \sum \Pr(s^t|s^{t-1}) \left[\begin{array}{c} u'\left(A_t L^\alpha - \frac{M(s^{t-1})}{d(s^{t-1})P(s^t)} - \frac{M(s^{t-1})}{P(s^t)}\right)\left(\alpha A_t L^{\alpha-1} - \frac{M(s^{t-1})/L}{d(s^{t-1})P(s^t)}\right) \\ +\frac{\xi(s^t)d(s^t)M(s^t)}{(1-d(s^t))L} \end{array} \right].
$$

$$
\frac{\xi(s^t)}{1-d(s^t)} = u'\left(A_t L^\alpha - \frac{M(s^{t-1})}{d(s^{t-1})P(s^t)} - \frac{M(s^{t-1})}{P(s^t)}\right)\frac{1}{P(s^t)}
$$

Eliminating $\xi(s^t)$ we obtain

$$0=\sum \Pr(s^t|s^{t-1})\left[\begin{array}{c} u'\left(A_tL^\alpha-\frac{M(s^{t-1})}{d(s^{t-1})P(s^t)}-\frac{M(s^{t-1})}{P(s^t)}\right) \\ \times\left(\alpha A_tL^{\alpha-1}-\frac{M(s^{t-1})}{d(s^{t-1})P(s^t)L}+\frac{d(s^t)M(s^t)}{P(s^t)L}\right)\end{array}\right].$$

Thus, we derive two dynamic equations

$$\begin{aligned}&\beta\sum \Pr(s^{t+1}|s^t)\left[u'\left(\frac{M(s^t)}{d(s^t)P(s^{t+1})}+\frac{M(s^t)}{P(s^{t+1})}\right)\frac{P(s^t)}{P(s^{t+1})}\right] \\ = \; & u'\left(A_tL^\alpha-\frac{M(s^{t-1})}{d(s^{t-1})P(s^t)}-\frac{M(s^{t-1})}{P(s^t)}\right)d(s^t),\end{aligned} \tag{D7}$$

$$0=\sum \Pr(s^t|s^{t-1})\left[\begin{array}{c} u'\left(A_tL^\alpha-\frac{M(s^{t-1})}{d(s^{t-1})P(s^t)}-\frac{M(s^{t-1})}{P(s^t)}\right) \\ \times\left(\alpha A_tL^{\alpha-1}-\frac{M(s^{t-1})}{d(s^{t-1})P(s^t)L}+\frac{d(s^t)M(s^t)}{P(s^t)L}\right)\end{array}\right]. \tag{D8}$$

In addition, the money market clearing condition must be satisfied:

$$\begin{aligned}\frac{M(s^{t-1})}{P(s^t)} &= \frac{M^d(s^t)}{P(s^t)}=A_tL^\alpha-\frac{B^d(s^{t-1})}{P(s^t)}-c^y(s^t) \\ &= A_tL^\alpha-\frac{M(s^{t-1})}{d(s^{t-1})P(s^t)}-c^y(s^t),\end{aligned}$$

and hence,

$$\frac{M(s^{t-1})}{P(s^t)}=\frac{A_tL^\alpha-c^y(s^t)}{1+\frac{1}{d(s^{t-1})}}.$$

Let $x(s^t)=\frac{M(s^{t-1})}{P(s^t)}=\frac{A_tL^\alpha-c^y(s^t)}{1+\frac{1}{d(s^{t-1})}}$. The money market clearing conditions imply the law of motion for the price level

$$\frac{P(s^t)}{P(s^{t+1})}=\frac{M(s^{t-1})}{M(s^t)}\frac{x(s^{t+1})}{x(s^t)}. \tag{D9}$$

Substituting (D9) into the dynamic equations (D7) and (D8) we get

$$\begin{aligned}&\beta\sum \Pr(s^{t+1}|s^t)\left[u'\left(\frac{x(s^{t+1})}{d(s^t)}+x(s^{t+1})\right)\frac{M(s^{t-1})}{M(s^t)}\frac{x(s^{t+1})}{x(s^t)}\right] \\ = \; & u'\left(A_tL^\alpha-\frac{x(s^t)}{d(s^{t-1})}-x(s^t)\right)d(s^t)\end{aligned}$$

$$0=\sum \Pr(s^t|s^{t-1})\left[\begin{array}{c} u'\left(A_tL^\alpha-\frac{x(s^t)}{d(s^{t-1})}-x(s^t)\right) \\ \times\left(\alpha A_tL^\alpha-\frac{x(s^t)}{d(s^{t-1})}+d(s^t)x(s^t)\frac{M(s^t)}{M(s^{t-1})}\right)\end{array}\right].$$

To close the system, we need to determine how $\frac{M(s^{t-1})}{M(s^t)}$ changes over time. Let $h(s^{t-1}) = \frac{M(s^{t-1})}{Z_t}$. Appendix B shows that

$$h(s^t) = \left(\frac{h(s^{t-1})}{x(s^t)}\right)^{\lambda} \exp(\sum \Pr(s^{t+1}|s^t) \left[\ln x(s^{t+1})\right]).$$

We end up with a system of $2S+1$ dynamic equations (D10)-(D12) that must be solved to compute the monetary equilibrium

$$\begin{aligned} & u'\left(A_t L^{\alpha} - \frac{x(s^t)}{d(s^{t-1})} - x(s^t)\right) d(s^t) \exp(\overline{\pi}) \\ = \ & \beta \sum \Pr(s^{t+1}|s^t) \left[u'\left(\frac{x(s^{t+1})}{d(s^t)} + x(s^{t+1})\right) \frac{h(s^{t-1})}{h(s^t)} \frac{x(s^{t+1})}{x(s^t)}\right], \text{ for all } s_t \end{aligned} \tag{D10}$$

$$0 = \sum \Pr(s^t|s^{t-1}) \left[\begin{array}{c} u'\left(A_t L^{\alpha} - \frac{x(s^t)}{d(s^{t-1})} - x(s^t)\right) \\ \times \left(\alpha A_t L^{\alpha} - \frac{x(s^t)}{d(s^{t-1})} + d(s^t) x(s^t) \frac{h(s^t)}{h(s^{t-1})} \exp(\overline{\pi})\right) \end{array} \right], \tag{D11}$$

$$h(s^t) = \left(\frac{h(s^{t-1})}{x(s^t)}\right)^{\lambda} \exp(\sum \Pr(s^{t+1}|s^t) \left[\ln x(s^{t+1})\right]), \text{ for all } s_t. \tag{D12}$$

where $x(s^t) = \frac{M(s^{t-1})}{P(s^t)}$ and $h(s^t) = \frac{M(s^t)}{Z_{t+1}}$.

Suppose we know $x(s^{t+1})$, $d(s^t)$. We can solve the above $2S+1$ equations for $2S+1$ unknown functions: $d(A_{t-1}, h(s^{t-1}))$, $x(A_t, A_{t-1}, h(s^{t-1}))$, and $h'(A_t, A_{t-1}, h(s^{t-1}))$. Note that in the case of i.i.d. productivity shocks these three functions do not depend on A_{t-1}. This is because the only way A_{t-1} enters the equations above, is through conditional probabilities $\Pr(s^t|s^{t-1})$. With i.i.d. shocks those are independent of s^{t-1}, and we are looking for $d(h(s^{t-1}))$, $x(A_t, h(s^{t-1}))$, and $h'(A_t, h(s^{t-1}))$.

In the CRRA utility case, the dynamic equations can be simplified somewhat

$$\begin{aligned} & \left(\frac{A_t L^{\alpha} - \left(\frac{1}{d(s^{t-1})} + 1\right) x(s^t)}{\frac{1}{d(s^t)} + 1}\right)^{-\gamma} d(s^t) x(s^t) \exp(\overline{\pi}) \\ = \ & \beta \frac{h(s^{t-1})}{h(s^t)} \sum \Pr(s^{t+1}|s^t) [x(s^{t+1})]^{1-\gamma}, \text{ for all } s_t \end{aligned}$$

$$0=\sum \Pr(s^t|s^{t-1})\left[\begin{array}{c}\left(A_tL^\alpha-\left(\frac{1}{d(s^{t-1})}+1\right)x(s^t)\right)^{-\gamma}\\ \times\left(\alpha A_tL^\alpha-\left(\frac{1}{d(s^{t-1})}-d(s^t)\frac{h(s^t)}{h(s^{t-1})}\exp(\overline{\pi})\right)x(s^t)\right)\end{array}\right],$$

$$h(s^t)=\left(\frac{h(s^{t-1})}{x(s^t)}\right)^\lambda \exp(\sum \Pr(s^{t+1}|s^t)\left[\ln x(s^{t+1})\right]),\text{ for all } s_t.$$

Appendix E: Optimal Monetary Policy in the Model With Land

The same welfare criterion as in the money only economy results in the following first-best allocations

$$
\begin{aligned}
c_t^{y*} &= \frac{1}{1+\beta^{\frac{1}{\gamma}}} A_t L^{\alpha}, \\
c_t^{o*} &= \frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}} A_t L^{\alpha}.
\end{aligned}
$$

The first-order conditions (D2) (D4) with the optimal allocations become:

$$
0 = \sum \Pr(s^t|s^{t-1}) \left[\begin{array}{c} \left(\frac{1}{1+\beta^{\frac{1}{\gamma}}} A_t L^{\alpha}\right)^{-\gamma} \left(\alpha A_t L^{\alpha-1} - \frac{M(s^{t-1})/L}{d(s^{t-1})P(s^t)}\right) + \\ +\beta \sum \Pr(s^{t+1}|s^t) \left[\left(\frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}} A_{t+1} L^{\alpha}\right)^{-\gamma} \frac{M(s^t)/L}{P(s^{t+1})} \right] \end{array} \right],
$$

$$
\begin{aligned}
0 &= \xi(s^t) - \left(\frac{1}{1+\beta^{\frac{1}{\gamma}}} A_t L^{\alpha}\right)^{-\gamma} \frac{1}{P(s^t)} \\
&\quad +\beta \sum \Pr(s^{t+1}|s^t) \left[\left(\frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}} A_{t+1} L^{\alpha}\right)^{-\gamma} \frac{1}{P(s^{t+1})} \right],
\end{aligned}
$$

$$
\frac{\xi(s^t)d(s^t)}{1-d(s^t)} = \beta \sum \Pr(s^{t+1}|s^t) \left[\left(\frac{\beta^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}} A_{t+1} L^{\alpha}\right)^{-\gamma} \frac{1}{P(s^{t+1})} \right].
$$

After simplifications, we obtain

$$
0 = \sum \Pr(s^t|s^{t-1}) \left[A_t^{-\gamma} \left(\alpha A_t L^{\alpha} - \frac{M(s^{t-1})}{d(s^{t-1})P(s^t)}\right) + \sum \Pr(s^{t+1}|s^t) \left\{ A_{t+1}^{-\gamma} \frac{M(s^t)}{P(s^{t+1})} \right\} \right],
$$

$$
d(s^t) = \sum \Pr(s^{t+1}|s^t) \left[\left(\frac{A_{t+1}}{A_t}\right)^{-\gamma} \frac{P(s^t)}{P(s^{t+1})} \right].
$$

From the money market clearing conditions

$$P(s^t) = \frac{M(s^{t-1})}{x(s^t)},$$

where the optimal $x(s^t)$ is given by

$$x(s^t) = \frac{\beta^{\frac{1}{\gamma}}}{\left(1+\beta^{\frac{1}{\gamma}}\right)\left(\frac{1}{d(s^{t-1})}+1\right)} A_t L^\alpha.$$

Substituting these values of $P(s^t)$, and $x(s^t)$, we obtain

$$0 = \sum \Pr(s^t|s^{t-1}) \left[\begin{array}{c} A_t^{-\gamma}\left(\alpha A_t L^\alpha - \frac{\beta^{\frac{1}{\gamma}} A_t L^\alpha}{\left(1+\beta^{\frac{1}{\gamma}}\right)\left(\frac{1}{d(s^{t-1})}+1\right)d(s^{t-1})}\right) \\ +\sum \Pr(s^{t+1}|s^t)\left\{A_{t+1}^{-\gamma}\frac{\beta^{\frac{1}{\gamma}}}{\left(1+\beta^{\frac{1}{\gamma}}\right)\left(\frac{1}{d(s^t)}+1\right)}A_{t+1}L^\alpha\right\} \end{array} \right],$$

$$d(s^t) = \sum \Pr(s^{t+1}|s^t)\left[\left(\frac{A_{t+1}}{A_t}\right)^{-\gamma}\frac{M(s^{t-1})}{M(s^t)}\frac{\frac{\beta^{\frac{1}{\gamma}}}{\left(1+\beta^{\frac{1}{\gamma}}\right)\left(\frac{1}{d(s^t)}+1\right)}A_{t+1}L^\alpha}{\frac{\beta^{\frac{1}{\gamma}}}{\left(1+\beta^{\frac{1}{\gamma}}\right)\left(\frac{1}{d(s^{t-1})}+1\right)}A_t L^\alpha}\right].$$

Which can be further simplified to

$$0 = \sum \Pr(s^t|s^{t-1}) \left[\begin{array}{c} A_t^{1-\gamma}\left(\alpha - \frac{\beta^{\frac{1}{\gamma}}}{\left(1+\beta^{\frac{1}{\gamma}}\right)\left(\frac{1}{d(s^{t-1})}+1\right)d(s^{t-1})}\right) \\ +\frac{\beta^{\frac{1}{\gamma}}}{\left(1+\beta^{\frac{1}{\gamma}}\right)\left(\frac{1}{d(s^t)}+1\right)}\sum \Pr(s^{t+1}|s^t)\left[A_{t+1}^{1-\gamma}\right] \end{array} \right],$$

$$d(s^t) = \sum \Pr(s^{t+1}|s^t)\left[\left(\frac{A_{t+1}}{A_t}\right)^{1-\gamma}\frac{M(s^{t-1})}{M(s^t)}\frac{\frac{1}{d(s^{t-1})}+1}{\frac{1}{d(s^t)}+1}\right].$$

For brevity, let's use the expectation operator

$$\left(1 - \frac{1}{\frac{1}{d(s^{t-1})}+1} - \frac{\alpha\left(1+\beta^{\frac{1}{\gamma}}\right)}{\beta^{\frac{1}{\gamma}}}\right)E_{t-1}\left[A_t^{1-\gamma}\right] = E_{t-1}\left[\frac{1}{\frac{1}{d(s^t)}+1}E_t\left[A_{t+1}^{1-\gamma}\right]\right], \qquad \text{(E1)}$$

$$\frac{M(s^t)}{M(s^{t-1})} = \frac{1}{d(s^t)} \frac{\frac{1}{d(s^{t-1})}+1}{\frac{1}{d(s^t)}+1} E_t \left[\left(\frac{A_{t+1}}{A_t} \right)^{1-\gamma} \right]. \tag{E2}$$

Let $\Psi(s^t) = \frac{1}{\frac{1}{d(s^t)}+1} E_t \left[A_{t+1}^{1-\gamma} \right]$. We can then rewrite (E1) as

$$\left(1 - \frac{\alpha \left(1 + \beta^{\frac{1}{\gamma}} \right)}{\beta^{\frac{1}{\gamma}}} \right) E_{t-1} \left[A_t^{1-\gamma} \right] = \Psi(s^{t-1}) + E_{t-1} \left[\Psi(s^t) \right]. \tag{E3}$$

Let us concentrate on stationary (first-best) equilibria, i.e. let $\Psi(s^t) = \Psi(s_t)$. The last equation is a functional equation which determines $\Psi(A_t)$. Once $\Psi(A_t)$ is known, we can use (E2) to solve for the optimal monetary policy. Optimal inflation is

$$\begin{aligned} \frac{P(s^{t+1})}{P(s^t)} &= \frac{M(s^t)}{M(s^{t-1})} \frac{x(s^t)}{x(s^{t+1})} \\ &= \frac{1}{d(s^t)} E_t \left[\left(\frac{A_{t+1}}{A_t} \right)^{1-\gamma} \right] \frac{A_t}{A_{t+1}}. \end{aligned}$$

Taking logs on both sides, and using properties of the log-normal distribution we obtain:

$$p(s^{t+1}) - p(s^t) = \ln \left(\frac{1}{d(s^t)} \right) + \frac{(1-\gamma)^2 \sigma^2}{2} + \gamma(1-\rho) a_t - \varepsilon_{t+1}.$$

It is easy to see from equation (E1) that the discount rate $d(s^t)$ is a constant in three special cases: first, when shocks to productivity are i.i.d. ($\rho = 0$), second, when shocks to productivity are a random walk ($\rho = 1$), and third, when utility is logarithmic ($\gamma = 1$). Let us prove this for an i.i.d. case.

When productivity shocks are i.i.d., equation (E1) simplifies to

$$1 - \frac{\alpha \left(1 + \beta^{\frac{1}{\gamma}} \right)}{\beta^{\frac{1}{\gamma}}} = \frac{1}{\frac{1}{d(s^{t-1})}+1} + E_{t-1} \left[\frac{1}{\frac{1}{d(s^t)}+1} \right]. \tag{E4}$$

Thus, we obtain two dynamic equations

$$\frac{M(s^t)}{M(s^{t-1})} = \frac{1}{d(s^t)} \frac{\frac{1}{d(s^{t-1})}+1}{\frac{1}{d(s^t)}+1} E_t \left[\left(\frac{A_{t+1}}{A_t} \right)^{1-\gamma} \right],$$

$$1 - \frac{\alpha\left(1+\beta^{\frac{1}{\gamma}}\right)}{\beta^{\frac{1}{\gamma}}} = \frac{1}{\frac{1}{d(s^{t-1})}+1} + E_{t-1}\left[\frac{1}{\frac{1}{d(s^t)}+1}\right]. \tag{E5}$$

Since we are focusing on stationary equilibria, assume $\frac{M(s^t)}{M(s^{t-1})} = \mu(s_t)$ and $d(s^t) = d(s_t)$. The first equation implies that $d(s_{t-1})$ is independent of s_{t-1}. Thus $d(s_t) = d$ for all s^t and for all t. Equation (E5) can be solved for d

$$d = \frac{\beta^{\frac{1}{\gamma}}(1-\alpha)-\alpha}{\beta^{\frac{1}{\gamma}}(1+\alpha)+\alpha}.$$